1 (7.2) Rotating flywheel on a rotating table

![Diagram of a flywheel on a rotating table]

As it can be seen from the picture the torque is out of the picture and its magnitude is:

$$\tau = 4T\alpha l,$$

(1)

where we assumed that the angle $\alpha$ is small. The derivative of the angular momentum $\frac{d\vec{L}_0}{dt}$ is also out the picture. That, and also its magnitude follows from:

$$\frac{d\vec{L}_0}{dt} = \vec{\Omega} \times \vec{L}_0$$

(2)

Thus $\tau = \frac{d\vec{L}_0}{dt} = \Omega L_0 = \Omega I_0 \omega_0$, and therefore:

$$\alpha = \frac{I_0 \omega_0 \Omega}{4Tl}.$$  

(3)

2 (7.5) Car on a curve

(a) Let us understand first why should the car tend to roll over. As one can see from the picture, $\vec{N}_1$, $\vec{f}_1$ and $\vec{f}_2$ create torque (about the center of mass), that’s into the picture, and $\vec{N}_2$ creates torque out of the picture. If the car is stable, then the total torque should be zero. By Newton’s second law:

$$f_1 + f_2 = Ma = \frac{Mv^2}{r}$$

$$N_1 + N_2 = Mg$$

(4)
Figure 2: View from behind. The car is turning to the left.

So the faster the car is moving the larger are $f_1$ and $f_2$, and thus the torque into the page. To have equilibrium, $N_1$ will have to decrease and $N_2$ will have increase. For high enough velocities $N_1$ will become zero and the car will start rolling over.

Now the reason putting a spinning flywheel can help is because we can put it in such way that the torque will be used to change the angular momentum of the flywheel and not of the car!

Now let’s consider the case where the car is turning to the left with angular velocity $\mathbf{\Omega}$ which will point vertically upward. Let’s put the flywheel so that its angular momentum $\mathbf{L}$ is pointing radially outward (to the right on the picture). To make that angular momentum rotate with the car, one needs torque equal to $dL/dt = \mathbf{\Omega} \times \mathbf{L}$, i.e. pointing forward with respect to the car, which is exactly what we’ve got.

Note, however, that if the car is turning to the right, we should not reverse the direction of the angular momentum, i.e. if the car is turning to the right the angular momentum should point radially inward. This is because in that case $\mathbf{\Omega}$ is pointing vertically downward, and the torque is pointing out of the page (backward w.r.t. the car). Thus the right choice for $\mathbf{L}$ would be pointing again to the right on the picture, so that $\mathbf{\Omega} \times \mathbf{L}$ is again in the direction of the torque. (More mathematically the reason the direction of $\mathbf{L}$ doesn’t change is because torque, angular velocity and angular momentum are axial vectors, not real vectors).

(b) If the loading on the wheels is the same, then $N_1 = N_2 \equiv N$, thus $f_1 = f_2 \equiv f$. Since we’ve already discussed the directions of the derivative of the angular momentum and the torque, let’s just write down the scalar version of the torque equation:

$$\tau = \frac{dL}{dt}. \tag{5}$$

Now

$$\tau = f_1 d \sin \alpha + N_1 d \cos \alpha + f_2 d \sin \alpha - N_2 d \cos \alpha = 2f d \sin \alpha = 2fL = Mv\mathbf{\Omega}L, \tag{6}$$

where in the last equality we used Newton’s second law \[\text{(4)}\]. Finally for a disk-shaped flywheel using

$$\frac{dL}{dt} = \mathbf{\Omega}L = \mathbf{\Omega}\mathbf{\omega} = \mathbf{\Omega} m R^2 \mathbf{\omega}, \tag{7}$$

we have:

$$\mathbf{\omega} = \frac{2MvL}{mR^2}. \tag{8}$$

3 (7.8) Deflecting hoop

(a) The torque induced by the force acting on the top of the hoop will be in the direction shown in the picture. Since $\mathbf{\tau} = \frac{dL}{dt}$, that means that the angular momentum will start rotating in the horizontal plane and if we look from the top it will be rotating counterclockwise ($\mathbf{\omega}$ pointing vertically upward). So in the gyroscope approximation the force will not incline the plane of the hoop, but will instead deflect the line of rolling.
By the conservation of momentum we have:

$$M \vec{v} + \vec{I} = M \vec{v}' ,$$

(9)

where $\vec{v}'$ is the velocity of the hoop after the tap. Thus the line of the rolling of the hoop will change by an angle:

$$\phi \approx \tan \phi = \frac{I}{Mv}. $$

(10)

(b) The gyroscope approximation is valid if $\frac{dL}{dt} \ll L\omega$, i.e. if the change in the angular momentum is small compared to the initial angular momentum. In our case this would mean that

$$\tau_{\text{max}} = Fb \ll L\omega = Mb^2\omega^2 = Mv^2. $$

(11)

Thus for the gyroscope approximation to hold, we need the peak applied force to satisfy:

$$F \ll \frac{Mv^2}{b}. $$

(12)

4 (10.2) Oscillating spring

For the damping constant we have:

$$\gamma = \frac{\omega_1}{Q} = \frac{2\pi \times 2 \text{Hz}}{60} \approx 0.21 \text{ s}^{-1}. $$

(13)

Then from

$$\omega_0 = \sqrt{\frac{k}{m}}, $$

(14)

we have:

$$k = m\omega_0^2 = m \left( \omega_1^2 + \frac{\gamma^2}{4} \right) = 0.3 \text{kg} \left( (2\pi \times 2 \text{Hz})^2 + \frac{(0.21 \text{ s}^{-1})^2}{4} \right) \approx 47.377 \text{ N/kg}. $$

(15)

5 (10.6) Falling masses and critical damping

(a) At the rest position we have equilibrium of forces, thus:

$$Mg = kx_0, $$

(16)

where $m$ is the mass of the platform and $x_0$ is the distance from the initial position of the platform to its final position.

Thus

$$k = \frac{Mg}{x_0} = 980 \text{ N/kg}. $$

(17)
(b) The velocity of the mass just before it hits the platform \( v \) is determined from energy conservation:

\[
Mgh = \frac{1}{2} Mv^2, \tag{18}
\]

where \( h \) is the height of the mass above the platform. Then we have an inelastic collision of the mass with the platform, so by momentum conservation:

\[
Mv = (M + m)v', \tag{19}
\]

where \( v' \) is their velocity just after they stick together.

Since we want critical damping we need \( \gamma = 2\omega_0 = 2\sqrt{\frac{k}{M+\ell}} \approx 18.07 \text{ s}^{-1} \).

Now the general equation of motion for critical damping is given by:

\[
x(t) = Ae^{-\frac{1}{2}t} + Bte^{-\frac{1}{2}t}. \tag{20}
\]

Here \( x(t) \) is the height of the platform w.r.t. the final position. Since we want \( x(0) = x_0 \) and \( \frac{dx}{dt}|_0 = v' \), we get for the coefficients \( A \) and \( B \):

\[
A = x_0
\]

\[
B = v' + \frac{\gamma}{2} = \frac{M}{M + m} \sqrt{2gh} + \sqrt{\frac{g}{x_0} \frac{M}{M + m}}. \tag{21}
\]

So finally the equation of motion is:

\[
x(t) = x_0 e^{-\sqrt{\frac{2}{x_0}} t} + \left( \frac{M}{M + m} \sqrt{2gh} + \sqrt{\frac{g}{x_0} \frac{M}{M + m}} \right) t e^{-\sqrt{\frac{2}{x_0}} t}. \tag{22}
\]