Physics 141 Homework #5
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414 A bead of mass \( m \) slides without friction on a smooth rod along the \( x \) axis. The rod is equidistant between two spheres of mass \( M \). The spheres are located at \( x=0 \), \( y=\pm a \) as shown, and attract the bead gravitationally.

a. Find the potential energy of the bead
b. The bead is released at \( x=3a \) with velocity \( V_0 \) toward origin. Find the speed as it passes the origin.
C. Find the frequency of small oscillations of the bead about the origin.

(A) The point here is that the net force exerted on the bead is along the \( x \) axis.

Its magnitude is:
\[
F = -2 \frac{G M m}{a^2 + x^2} \frac{x}{\sqrt{a^2 + x^2}}
\]
It’s twice of the \( x \) component of the force that a single \( M \) applies on \( m \). The minus sign indicates that the direction of the net force is always pointing toward the origin from \( m \).

The potential energy: \( U_{\infty} - U(x) = -\int_{x}^{\infty} F dx' \), we define \( U_{\infty} = 0 \)
\[
U(x) = \int_{x}^{\infty} F dx' = -G M m \left[ -\frac{x^2}{2(a^2 + x^2)} \right]^{x}_{\infty} = 2 G M m \frac{1}{\sqrt{a^2 + x^2}} \bigg|_{x}^{\infty}
\]
\[
= -\frac{2 G M m}{\sqrt{a^2 + x^2}}
\]
4.14 (a) If you know the superposition of energy, you can do it in this way:

The potential energy of the bead comes from the interaction with the two spheres. Each one leads to the potential energy:

\[ U(x) = -\frac{GMm}{\sqrt{a^2 + x^2}} \]

i.e.

The potential energy is directly given by:

\[ U(x) = -2 \frac{GMm}{\sqrt{a^2 + x^2}} \]
(b) By the conservation of energy, and suppose the speed as it passes the origin is \( V \),

\[
U(3a) + \frac{1}{2} m v_0^2 = U(0) + \frac{1}{2} m V^2
\]

\[
U(x) = -\frac{2GMm}{\sqrt{a^2 + x^2}} \quad \therefore \quad -\frac{2GMm}{\sqrt{a^2}} + \frac{1}{2} m v_0^2 = -\frac{2GMm}{a} + \frac{1}{2} m V^2
\]

\[
V = \sqrt{v_0^2 + \frac{4GM}{a} \left(1 - \frac{\sqrt{a}}{v_0}\right)}
\]

(c) Our force has the form of:

\[
F = -2 \frac{GMm}{(a^2 + x^2)^{3/2}} x
\]

Around the origin, it suggests that \( F \) is proportional to \( x \):

\[
F \approx -2 \frac{GMm}{a^3} x, \quad \text{when } x \to 0
\]

(If you want the detail, you can expand \( F \) in Taylor series.)

\[
k = \frac{2GMm}{a^3}
\]

\[
\therefore \text{ our frequency is } \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{\pi} \sqrt{\frac{GM}{2a^3}}
\]

4.18 A 160-lb man leaps into the air from a crouching position. His center of gravity rises 1.5 ft before he leaves the ground, and it then rises 3 ft to the top of his leap. What power does he develop assuming that he pushes the ground with constant force?

The ground will also exert the equal force \( F \) to the person. By the work-energy theorem,

\[
(F - mg) \cdot 1.5 \text{ ft} = \frac{1}{2} mg \cdot 3 \text{ ft}
\]
Because the man is at rest before the leap and at the top, we get: \( F = 3mg \)

The power he develops will reach the peak when he leaves the ground. At this point, the speed reaches the peak. It's the initial speed of the man during the motion after he leaves the ground.

\[ V = \sqrt{2g \cdot 3ft} \]

\[ P_{\text{max}} = F \cdot V = 3mg \sqrt{2g \cdot 3ft} = 3 \times 160 \times 4.45 \times \sqrt{2 \times 9.8 \times 3 \times 0.305} \quad W = 4.05 \times 10^2 \ W 
\]

\[ P_{\text{max}} > 10 \text{ hp} = 7.46 \times 10^3 \ W 
\]

The problem is asking about the average power.

Again from the constant force, we get the time consumed in the first 1.5 ft as, (Average speed equals to half of the final speed,)

\[ t = \frac{1.5 \text{ ft}}{V} = \frac{3 \text{ ft}}{\sqrt{2g \cdot 3ft}} = \frac{3 \times 0.305}{\sqrt{2 \times 9.8 \times 3 \times 0.305}} \quad s = 0.216 \ s 
\]

The work during the motion by the man is:

\[ W \cdot 1.5 \text{ ft} = mg \cdot 3 \text{ ft} + mg \cdot 1.5 \text{ ft} = mg \cdot 4.5 \text{ ft} \]

\[ \bar{P} = \frac{mg \cdot 4.5 \text{ ft}}{t} = \frac{160 \times 4.45 \times 4.5 \times 0.303}{0.216} \quad W = 4.49 \times 10^3 \ W \quad 1 \text{ hp} \bar{P} < 10 \text{ hp} 
\]

4.19 The man in the preceding problem again leaps into the air, but this time the force he applies decreases from a maximum at the beginning of the leap to zero at the moment he leaves the ground. As a reasonable approximation, take the force to be \( F = F_0 \cos(wt) \), where \( F_0 \) is the peak force, and contact with the ground ends when \( wt = \frac{\pi}{2} \). Find the peak power the man develops during this jump.
According to the relation between impulse and momentum, we get \( V(t) \):

\[
V(t) = \frac{1}{m} \int_0^t (F_0 \cos \omega t' - mg) \, dt' = \frac{F_0}{m} \sin \omega t - gt
\]

The power at time \( t \) is:

\[
P(t) = F \, V(t) = \frac{F_0^2}{2m} \sin 2\omega t - F_0 g t \cos \omega t
\]

It will be impractical to find the peak power by \( \frac{dp(t)}{dt} = 0 \).

An reasonable approximation is: \( F_0 \gg mg \)

Drop the second item in \( p(t) \), we get:

\[
P(t) = \frac{F_0^2}{2m} \sin 2\omega t
\]

\[
\frac{dp(t)}{dt} = \frac{F_0^2}{2m} 2 \omega \cos 2\omega t = \frac{F_0^2}{m} \cos 2\omega t = 0
\]

we get \( \omega t = \frac{\pi}{4} \), it's the half time

\[
P_{\text{max}} = \frac{F_0^2}{2m} \sin 2 \cdot \frac{\pi}{4} = \frac{F_0^2}{2m}
\]

4.20 Sand runs from a hopper at constant rate \( dm/dt \) onto a horizontal conveyor belt driven at constant speed \( V \) by a motor.

(a) Find the power needed to drive the belt.

(b) Compare the answer to (a) with the rate of change of kinetic energy of the sand. Can you account for the difference?

(a) The belt will exert push the sand of mass \( dm \) to give it a momentum of \( V dm \) in \( dt \). Suppose the force is \( F \),

\[
F \, dt = dm \, V \quad \therefore F = V \frac{dm}{dt}
\]

\[
\therefore \text{The power is:} \quad P = F \, V = V^2 \frac{dm}{dt}
\]
(b) The rate of change of kinetic energy of the sand is:
\[ \frac{1}{2} \frac{d(mv^2)}{dt} = \frac{1}{2} v^2 \frac{dm}{dt} \]

It's not equal to our power.

The difference comes from the friction and the relative motion between the belt and the sand. When the sand just get into touch with the belt with no speed, it's accelerated by the friction. So at first, the sand is slower than the belt. Mechanical Energy is transformed into heat in this way.

4.25 A proton makes a head-on collision with an unknown particle at rest. The proton rebounds straight back with \( \frac{4}{9} \) of its initial kinetic energy. Find the ratio of the mass of the unknown particle to the mass of the proton, assuming that the collision is elastic.

By Conservation Laws, we get:

\[ m_p V_{p_2} + m V_2 = m_p V_p \]  \hspace{1cm} (1)

\[ \frac{1}{2} m_p V_{p_2}^2 + \frac{1}{2} m V_2^2 = \frac{1}{2} m_p V_p^2 \]  \hspace{1cm} (2)

\( m_p \), mass of the proton  \( m \), mass of the unknown particle.

\[ \frac{1}{2} m_p V_{p_2}^2 = \frac{4}{9} \cdot \frac{1}{2} m_p V_p^2 \]

\[ \therefore V_{p_2} = -\frac{2}{3} V_p \]  \hspace{1cm} (Because it's bounced back.)

From the conservation of momentum, we get,

\[ \frac{5}{3} m_p V_p = m V_2 \]

Continue to use (2): \( m V_2^2 = \frac{5}{9} m_p V_p^2 \)

\[ \therefore V_2 = \frac{1}{3} V_p, \quad \text{and} \quad m = 5 m_p \quad \frac{m}{m_p} = 5 \quad \text{this is the ratio} \]
4.26° A particle of mass $m$ and initial velocity $V_0$ collides elastically with a particle of unknown mass $M$ coming from the opposite direction as shown in the sketch below. After the collision $m$ has velocity $V_{0/2}$ at right angles to the incident direction, and $M$ moves off in the direction shown in the sketch. Find the ratio $M/m$.

Suppose the velocities of $M$ before and after the collision are $V_1$ and $V_2$ separately.

Write down the conservation laws as:

\[ mV_{0/2} = MV_2 \sin 45° = \frac{\sqrt{2}}{2} MV_2 \quad \text{①} \]
\[ MV_0 - MV_1 = \frac{\sqrt{2}}{2} MV_2 \quad \text{②} \]
\[ \frac{1}{2} mV_0^2 + \frac{1}{2} MV_1^2 = \frac{1}{2} m\left(\frac{V_0}{2}\right)^2 + \frac{1}{2} MV_2^2 \quad \text{③} \]

From ① and ③, we get: 

\[ MV_0 - MV_1 = mV_{0/2} \]

\[ \therefore V_1 = \frac{m}{2M} V_0 \]

By ①, we have $V_2$ as: 

\[ V_2 = \frac{\sqrt{2}m}{2M} V_0 \]

Plug them into ②:

\[ \frac{1}{2} mV_0^2 + \frac{1}{2} M\left(\frac{m}{2M} V_0\right)^2 = \frac{1}{2} m\left(\frac{V_0}{2}\right)^2 + \frac{1}{2} M\left(\frac{\sqrt{2}m}{2M} V_0\right)^2 \]

It can be simplified as:

\[ \frac{1}{2} + \frac{1}{8} \frac{m}{M} = \frac{1}{8} + \frac{1}{4} \frac{m}{M} \]

\[ \therefore \frac{m}{M} = 3 \]

\[ \frac{M}{m} = \frac{1}{3} \]